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PROCEEDINGS
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THE PREPARATION OF SPECIFIC ADSORBENTS

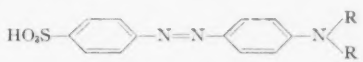
By Frank H. Dickey*

Department of Chemistry, California Institute of Technology†

Communicated by L. Pauling, March 21, 1949

A method has been developed for preparing adsorbents having specific affinities for predetermined substances. This method consists in forming the structure of the adsorbent in the presence of the particular compound for which it is desired to prepare a specific adsorbent. The presumable explanation of the formation of a specifically attracting structure under these conditions is that the adsorbent in the process of formation has accessible to it a very great number of structures which differ only slightly in stability, and that in the presence of a foreign molecule those structures that are stabilized through attraction for the foreign molecule are preferentially assumed. The adsorbent is thus pictured as automatically forming pockets that fit closely enough to the foreign molecule to hold it by van der Waals' forces, hydrogen bonds, interionic attractions, and other types of intermolecular interaction. This mechanism is the same as that proposed by Pauling[1] for the formation of antibodies with use of antigen molecules as a template, which formed the basis for the manufacture of artificial antibodies reported by Pauling and Campbell.[2]

An experimental test of the idea has been made by the formation of silica gel in the presence of methyl orange and some of its homologs—ethyl, *n*-propyl and *n*-butyl orange. These compounds have the following structure:



where *R* represents the radicals named. The adsorbents were prepared by mixing 0.5 g. of finely divided dye with 30 ml. of aqueous sodium silicate (d^{20} 1.401) 275 ml. of water and 30 ml. of glacial acetic acid, drying (at room temperature), grinding and sieving the resultant gels, and subjecting the fraction between 48 and 200 mesh to methanol extraction, which removed most of the dye. Control gels, without dye, were prepared in the same way.

The gels were tested as adsorbents by bringing 1-g. samples into equilibrium with 10-g. portions of standard solutions of the dyes in 5% acetic acid. The initial dye concentrations were 1.5×10^{-5} formal. The final concentrations in the supernatant solutions were measured with a spectrophotometer and the quantity adsorbed was calculated by difference. In order to obtain accurate estimates of the dye concentrations from the spectrophotometer readings it was necessary to correct these readings for the cloudiness which appeared in the solutions after contact with the silica gel samples. Such corrections were calculated on an empirical basis from readings taken in another portion of the spectrum where the dye absorptions are negligible. The ratio of the concentration in the gel to that in the supernatant solution was then calculated as a measure of the "adsorption power" of the gel for the particular dye. The following results were obtained.

TABLE 1

	ADSORPTION POWER FOR:			
	METHYL ORANGE	ETHYL ORANGE	PROPYL ORANGE	BUTYL ORANGE
Control gel	84	80	240	320
Specific adsorbent	300	740	5000	5000

In order to show that this process has effected a specific rather than a general enhancement of the adsorption properties of the gels, adsorption powers were determined with each of the gels for each of the dyes. The ratio of these adsorption powers to the adsorption powers shown for the particular dyes by the control gels were then calculated. These "relative adsorption powers" are tabulated below.

TABLE 2

	RELATIVE ADSORPTION POWER FOR:			
	METHYL ORANGE	ETHYL ORANGE	PROPYL ORANGE	BUTYL ORANGE
Gel prepared with:				
Methyl orange	3.5	1.6	1.1	1.1
Ethyl orange	2.5	9	2.1	2.2
Propyl orange	2.3	5	20	6
Butyl orange	1.5	2.8	5	15

It is difficult to reproduce exactly the adsorption properties of silica gel samples and these data are to be regarded primarily as a qualitative demonstration of the preparation of specific adsorbents. However, the essential features of the above table, maximum adsorption power for the dye used in the preparation of the adsorbent, and sharply decreasing adsorption powers for the dyes of decreasing similarity of molecular structure, have been observed in several independent experiments. Gels prepared with this series of compounds are not better adsorbents than the control gels for a struc-

turally unrelated dye, fluorescein. The results obtained with this series of compounds are especially striking because the dyes differ only in the nature of the alkyl groups. Still more pronounced effects would be expected in studies of compounds having greater structural differences.

Many obvious applications and extensions of these ideas are now being investigated and detailed results will be published later. Of particular interest are an investigation of the possibility that optical isomers can be separated with such adsorbents and an attempt to demonstrate "specific catalysis," analogous to the action of enzymes, by specific adsorbents for the reactants or products of various reactions.

* E. I. du Pont Predoctoral Fellow in Chemistry.

† Contribution No. 1282 from the Gates and Crellin Laboratories of Chemistry, California Institute of Technology, Pasadena, California.

[1] Pauling, L., *J. Am. Chem. Soc.*, **62**, 2643–2657 (1940).

[2] Pauling, L., and Campbell, D. H., *J. Exptl. Med.*, **76**, 211–220 (1942).

THE VALENCE-STATE ENERGY OF THE BIVALENT OXYGEN ATOM

BY LINUS PAULING

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Communicated March 21, 1949

A significant simplification in the discussion of the structure of molecules can be achieved by referring the energies of molecules not to the energies of their constituent atoms in any one of the stationary spectroscopic states of the isolated atoms, but instead to a hypothetical valence state of each atom.[1] The valence state of an atom is defined as that state in which it has the same electronic structure as it has in the molecule. Thus in the valence state the spins of the electrons that are used for forming bonds in a molecule are considered to resonate between positive and negative values, corresponding to the resonance of each bonding electron with another electron of opposite spin in the atom with which the bond is formed. For the bivalent oxygen atom, with six L electrons, two orbitals, one mainly $2s$ in character and one $2p$, are occupied by unshared pairs, and the other two orbitals, mainly $2p$ but with a small amount of s hybridization, are occupied by one bonding electron apiece, the spins of which are parallel 50 per cent of the time and antiparallel 50 per cent of the time. In the following paragraphs values of the energy of this bivalent state of the oxygen atom relative to the normal state of the isolated atom, ^3P, are derived by four different methods. It is found that the four values agree

reasonably well with one another, and their average value, 0.74 v.e., is indicated to be reliable to about 0.05 v.e.

According to the Slater theory, the energy values of the three Russell-Saunders states based upon the normal configuration of the oxygen atom are $F^0 - \frac{5}{25} F^2$, $F^0 + \frac{1}{25} F^2$, and $F^0 + \frac{10}{25} F^2$, for ^3P, ^1D, and ^1S, respectively. The spectrum of oxygen shows that the ^1D state lies 9239 cm.$^{-1}$ above ^3P and ^1S 9739 cm.$^{-1}$ above ^1D. These differences in energy are not in the ratio 2:3 predicted by simple spectroscopic theory. For the valence state of the atom, neglecting hybridization, the energy is predicted to be equal to the Coulomb interaction of two p electrons occupying different orbitals directed along two mutually orthogonal axes, $F^0 - \frac{2}{25} F^2$, and one-half of the resonance integral between these two orbitals, that is, $-\frac{3}{50} F^2$, a total of $F^0 - \frac{7}{50} F^2$. F^2 is indicated to have the value 6.16 v.e. by the $^1S - {}^1D$ separation, and 8.21 v.e. by the $^1D - {}^3P$ separation. If the ^3P state is considered to be depressed in energy through resonance with a ^3P state based on a higher configuration, then the energy of the valence state relative to ^3P is predicted to be 0.86 v.e., which is the energy of ^1D corrected by the quantity $-\frac{9}{50} F^2$ with F^2 given by the $^1S - {}^1D$ separation. Similarly if ^1S is considered to be depressed by resonance, the energy of the valence state is predicted to be 0.49 v.e. In the absence of knowledge about the nature of the cause of the deviation of the intervals from the theoretical ratio the best that we can do is to take the average, 0.68 v.e., as the spectroscopic value of the energy of the valence state.[2] The true value would be expected to be slightly higher because of hybridization of the bond orbitals; s-p hybridization in the alkali-metal diatomic molecules gives rise to a valence-state energy of 0.1 to 0.2 v.e. for the alkali atoms,[3] and similar effects of hybridization may be expected for other atoms.

A second, independent value of this energy quantity can be obtained by use of the postulate that the Morse function represents correctly the bond energy of two atoms in their valence states, and that the energy of a molecule obtained by extrapolation from the low-lying vibrational energy levels corresponds to the valence states of the atoms in the molecule. Linear extrapolation of the vibrational energy levels of the normal state of the oxygen molecule, $O_2 X^3\Sigma_g^-$, with use of the values 1580 cm.$^{-1}$ for ω_e and 11.35 cm.$^{-1}$ for $\omega_e x_e$ (as given by the slope of the best straight line for $\Delta G(v + {}^1/_2)$ to $v = 17$), leads to 6.69 v.e. for the energy of dissociation to two oxygen atoms in the valence state. The dissociation energy of the normal oxygen molecule into oxygen atoms in the normal state is 5.084 v.e., and in conse-

quence the energy of the valence state of the oxygen atom is indicated to be 0.80 v.e., one-half the difference of these two quantities.

A similar calculation can be made for the OH molecule. This molecule is to be considered as containing a bivalent oxygen atom with one valence free. The extrapolated energy of dissociation into a hydrogen atom and an oxygen atom in its valence state is 5.03 v.e., corresponding to the observed values $\omega_e = 3735$ cm.$^{-1}$ and $\omega_e x_e = 82.6$ cm.$^{-1}$. The dissociation energy into a hydrogen atom and a normal oxygen atom is 4.34 v.e., as given by the work of Dwyer and Oldenberg.[4] The difference between these values, 0.69 v.e., is the energy of the valence state of oxygen in this molecule.

It is very interesting to note that the difference between the energy of removal of the first hydrogen atom from a water molecule, involving breaking one oxygen-hydrogen bond, and the energy of removal of the second hydrogen atom vanishes when the valence state of oxygen is used as the point of reference. The energy of removal of the first hydrogen atom is found by combining thermochemical data with Dwyer and Oldenberg's value for the dissociation energy of OH to be 5.12 v.e., whereas that for the second hydrogen atom is 4.34, relative to the normal state of the oxygen atom. If the valence state of oxygen is used as a point of reference, the second of these values is to be increased by the energy of the valence state, which has been shown above to be approximately 0.7 v.e., and it is thus brought into approximate equality with the first bond energy. The equality of these two energy quantities to within 2 per cent is a striking illustration of the significance of the chemist's picture of the molecule as involving chemical bonds between pairs of atoms.

If the assumption is made that in this molecule the energy of breaking the first bond is exactly equal to the energy of breaking the second bond, when the valence state of oxygen is taken as the point of reference, an independent value can be derived for the energy of the valence state of oxygen. This value, 0.78 v.e., the difference of the two dissociation energies, agrees reasonably well with the others.

The agreement between the values of the valence-state energy of the oxygen atom in three different molecules, O_2, OH, and H_2O, suggests that the same value will hold for other molecules also, and this suggestion is supported by the agreement with the range of values given by the simple theory of atomic spectroscopic states. The average of the four independent values of the energy of the bivalent state of the oxygen atom relative to the normal spectroscopic state, 0.74 v.e. or 17.1 kcal./mole, is presumably reliable to about 0.05 v.e.

* Contribution No. 1283 from the Gates and Crellin Laboratories of Chemistry.

[1] Pauling, L., *Sommerfeld Festschrift, Z. Naturforsch.*, **3a**, 438–447 (1948).

[2] In the discussion of the valence state of oxygen in reference 1 the energy of the valence state was given incorrectly as $F^0 - \frac{11}{100} F^2$.

[3] Pauling, L., *Proc. Roy. Soc., London*, **A196**, 343–362 (1949).

[4] Dwyer, R. J., and Oldenberg, O., *J. Chem. Phys.*, **12**, 351–361 (1944).

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*ACQUISITION OF THE J SUBSTANCE BY THE BOVINE ERYTHROCYTE**

BY CLYDE STORMONT†

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Communicated by R. A. Brink, April 2, 1949

The antigenic pattern of the bovine erythrocyte is genetically conditioned by numerous genes on at least nine of the thirty pairs of chromosomes.[1, 2] The combinations of these genes permit well over a million blood types. Any two animals taken at random, therefore, will almost invariably possess distinct blood types. A study of bovine twins, however, disclosed a much higher proportion of identical blood types than is expected in view of the relatively low frequency of monozygotous twinning in cattle.[3] Owen pointed out that this high proportion of identical blood types could be accounted for as a consequence of the known frequent union of chorionic blood vessels of twin fetuses, as demonstrated by Lillie.[4] Apparently, embryonal blood cells are interchanged by those twins having a common circulatory system and these primordial cells settle in the hematopoietic tissues where they serve as a source of erythrocytes throughout the life of each twin. The identity of blood types in otherwise genetically dissimilar twins would thus result from a mixture of two kinds of erythrocytes, namely, those produced by (*a*) the twin's own cells and (*b*) cells transplanted from the co-twin. Several lines of evidence, including the *in vitro* separation of the two kinds of erythrocytes comprising the mosaic of blood cells, afford proof of this explanation. It is now found that the cellular character called J constitutes an exception to the general rule in that twins possessing a blood admixture nevertheless may differ in respect to the presence or absence of this blood factor. The present report is concerned with the reasons for this non-conformity in behavior of character J.

Each of the numerous antigenic factors of cattle erythrocytes is recognized by the reaction (hemolysis) of these cells produced by a corresponding reagent (antibodies) and complement (fresh rabbit serum). For example, erythrocytes of an individual that possesses a gene for antigen A are sensitized by the action of A reagent and subsequently destroyed (hemolysed) by the action of the complement, while those of an animal that lacks gene A

are non-reactive in the presence of A reagent and complement. The blood cells of an individual are tested with each of 40-odd reagents for the presence or absence of reactivity (hemolysis). The blood type (antigenic formula) of an animal, as determined by these tests, constitutes the reactions of the cells when tested with each of the reagents. A diagrammatic representation of the techniques involved and references to previous reports are given by Stormont and Cumley.[5]

TABLE 1

BLOOD ADMIXTURE IN A PAIR OF TWINS DIFFERING IN FACTOR J

	BLOOD TYPES													
Twin no. 1:	A	$C_1^{\pm}$	F	$G^{\pm}$	H	J	P	$Q^{\pm}$	$V^{\pm}$	W	X_2	Z	$D'^{\pm}$	H′	I′
Twin no. 2:	A	$C_1^{\pm}$	F	$G^{\pm}$	H		P	$Q^{\pm}$	$V^{\pm}$	W	X_2	Z	$D'^{\pm}$	H′	I′
	BLOOD TYPES OF THE SEPARATED CELLS FROM TWIN NO. 1													
Type 1:	A	C_1	F		H	J	P	Q	V	W	X_2	Z		H′	I′
Type 2:	A		F	G	H	J	P			W	X_2	Z	D′	H′	I′
	BLOOD TYPES OF THE SEPARATED CELLS FROM TWIN NO. 2													
Type 1*a*:	A	C_1	F		H		P	Q	V	W	X_2	Z		H′	I′
Type 2*a*:	A		F	G	H		P			W	X_2	Z	D′	H′	I′

Generally, erythrocytes that are sensitized with sufficient antibody are completely hemolysed when complement is added. The cells of the twins (table 1), however, exhibited only partial hemolysis when sensitized with antibodies for the factors C, G, Q, V and D′, as indicated by the ± sign above these factors. Cell counts were made following the hemolysis of aliquots of 2.5% suspensions of the erythrocytes of each twin. These counts showed that approximately 65% of the cells of each twin were hemolysed following sensitization with either the G or D′ reagents, whereas about 35% were destroyed following sensitization with the reagents for C, Q, and V, respectively. These results demonstrated that only one kind of erythrocytes in the mixture possessed the antigenic factors G and D′ while the other kind possessed the factors C, Q and V not found in the first. The cells that were not hemolysed following sensitization with the G (or D′) reagent contained the factors C, Q and V as well as the factors common to both types of blood (i.e., A, F, H, P, W, X_2, Z, H′ and I′). Likewise, the other kind of cells, which were not lysed by the C reagent (or by those for Q or V), contained the factors G and D′ in addition to those common to both kinds of cells. The J character obviously did not conform to the behavior of the others, since it appeared in both kinds of the blood cells of twin no. 1 and was not demonstrable in either kind of the cells of the cotwin.

The J character is inherited as a dominant in contrast with its absence,[6] as are the other known antigenic components of the erythrocytes of cattle.

It would appear, however, that erythrocytes derived from the primordial blood cells of an individual possessing the gene for J (twin no. 1, table 1) fail to manifest this phenotype in a co-twin-host lacking the J gene. Conversely, those erythrocytes derived from an individual lacking the gene are capable of developing this character in a host possessing the gene.

The unique behavior of J in individuals of multiple births focused attention on other peculiarities of this blood factor that suggested a plausible explanation of this phenomenon.

1. The J character has been detected only by means of normal antibodies present in certain cattle that lack J.[6] Attempts to produce immune antibodies specific for J in cattle and rabbits have not been successful to date. In contrast, all the other blood factors of cattle are recognized by means of immune antibodies.

2. The J character of the erythrocytes seemingly is not present at birth, since the cells of several young calves which originally were J-negative were found in subsequent tests to be J-positive.

3. A soluble J or J-like substance has been found in considerable concentrations in the plasma of J-positive individuals. (This observation was made by Mrs. M. W. Yeas, formerly of this laboratory, who noted that the plasma of J-positive bloods was capable of inhibiting the reaction of J antibodies.) Soluble substances corresponding to the other blood-group-factors of cattle erythrocytes have not been detected.

4. Marked differences in the avidity or reactivity of the cells of different J-positive adults for the same concentrations of J antibodies have been noted. However, the degree of reactivity of the cells of a given individual appears to be relatively constant from month to month. Qualitative differences in the J substance have not been clearly demonstrable by antibody-absorption. On the other hand, the titer of J antibodies varies markedly among individuals and within the same individual from one time to another. As a consequence of such differences, weakly reactive anti-J sera do not contain sufficient antibody to cause visible hemolysis of all J-positive bloods. A similar effect may be obtained by diluting the antibodies of a more strongly reactive normal serum. Two potent normal sera (titer 1:256 and 1:512, respectively) obtained from the same cow (No. 17Z) were used for the tests described in this report. These sera were employed routinely in optimum dilution (1:4 or 1:8) for the detection of the cellular character J.

These observations suggested the explanation that J is primarily a soluble, serologically reactive substance which may be acquired by the erythrocytes of any individual (with or without the gene) on contact with blood plasma containing the soluble substance. To test this assumption, the bloods of donors lacking J, whose cells could be separated and readily

distinguished from those of the recipients, were respectively transfused into selected recipients, as given in table 2.

The cells of donor 41 that were recovered from a blood sample taken from the recipient (cow no. Z21, table 2) just after the transfusion were J-negative. The cells from each of the donors (cows 41 and Z26) recovered at intervals of 2, 4, 7, 9, 14 and 21 days after transfusion of 2.5 liters of whole citrated blood had become J-positive. The cells recovered after the fourth day were indistinguishable from those of the recipients in their degree of reactivity with J antibodies.

The results suggested that the same transformation might be accomplished *in vitro*, if given sufficient time. Washed erythrocytes from J-negative cattle were suspended in freshly filtered (Seitz filter) plasma of J-positive individuals (approximately 0.4 ml. of cells to 10 ml. of plasma) and incubated at 37°C. The cells were changed to fresh plasma at intervals of 2 hours during the day and were kept at refrigerator temperature (±4° C.)

TABLE 2

THE *in vivo* TRANSFORMATION OF J-NEGATIVE ERYTHROCYTES TO J-POSITIVE

EXPERIMENT NO. 1	BLOOD TYPES										
Donor 41		C_1	F	G			X_2	Y_2	Z	E'_1	H′
Recipient Z21	A	C_1	F	G	*J*	W	X_2	Y_2	Z	E'_1	H′
EXPERIMENT NO. 2											
Donor Z26	A	C_1	F			O_3	S	D′		E'_3	H′
Recipient Z18	A	C_1	F	G	*J*	O_3	S	Z D′		E'_4	H′
BLOOD TYPES OF DONORS' CELLS RECOVERED 2 DAYS, AND AT LATER INTERVALS POST-TRANSFUSION											
Donor 41		C_1	F	G	*J*		X_2	Y_2	Z	E'_1	H′
Donor Z26	A	C_1	F		*J*	O_3	S	D′		E'_3	H′

without a further change overnight. The treated cells became J-positive within 24 hours. These experiments are in agreement with the proposal that the J character of cattle is primarily a soluble, serologically reactive substance that is acquired by erythrocytes in contact with this substance in the blood plasma.

Assuming that tissues which produce soluble J substance, or the cells ancestral to these tissues, are not interchanged by fetuses during embryonic life, the behavior of J in twins having mixed blood types is readily understood. Both kinds of erythrocytes in a twin possessing the gene acquire the soluble substance from the plasma, whereas both kinds of cells of the co-twin that lacks the gene are J-negative.

Holtman[7, 8] demonstrated that bacteria are capable of acquiring serologically reactive substances from culture-media and from the animal host. The acquired substances were from media containing horse serum and from guinea pig hosts. Both these species possess Forssman antigen as well as serologically reactive substances related to A of man. As Holtman pointed

out, certain of the heterophile reactions attributed to bacterial cells may be consequent on the acquisition of these substances from the media or the host.

A definite serological relationship between the J substance of cattle and the A of humans was also established. Normal anti-J sera, that did not contain appreciable agglutinins for the cells of humans of groups O and B, nevertheless agglutinated, in higher titers, the cells of individuals of blood groups A and AB. Moreover, both the saliva and sera of humans of groups A and AB inhibited J antibodies in their reaction with J cells. Presumably, this was due to the presence of a soluble, serologically reactive substance in the saliva and serum corresponding to the A substance of the cells. Likewise, commercial preparations of pepsin and trypsin contained a soluble, heat-stable substance (or substances) which was capable of inhibiting J antibodies. Presumably the J substance is closely allied serologically and thereby chemically with other A or A-like polysaccharides generally distributed throughout the animal kingdom.

With few exceptions, the antigens of blood cells have behaved genetically as simple dominants in contrast to their absence. Moreover, their relative ease of detection by means of antibodies, their apparent constancy under varied environmental conditions and throughout life as well as the fact that these specific properties are contained in the stroma of erythrocytes[9] would suggest that they are produced by the action of genes in maturing erythrocytes or their cellular precursors. Indeed, the idea has been expressed[10, 11] that cellular antigens may be the more or less primary products of their causative genes. While the evidence concerning the J factor of cattle does not constitute an exception, as far as is known, to the above idea, it demonstrates that the synthesis may not always be in the cells in which a particular factor is recognized.

Summary.— Evidence is presented that the J factor of bovine erythrocytes is primarily a soluble, serologically reactive substance elaborated by tissues other than those which produce erythrocytes. Erythrocytes acquire the J substance on contact with soluble J in the plasma.

* Appreciated contributions to various phases of this study have been made by Professor M. R. Irwin. Considerable technical assistance to the project was rendered by Miss M. Claire Busk.

† From the Departments of Genetics (No. 396), Veterinary Science and Dairy Husbandry, University of Wisconsin, in coöperation with the Bureau of Animal Industry, U. S. Department of Agriculture. This study was supported in part by the Research Committee of the Graduate School from funds supplied by the Wisconsin Alumni Research Foundation. Blood samples were obtained through the generous coöperation of workers at state experiment stations and of numerous private breeders of cattle.

[1] Stormont, C., Owen, R. D., and Irwin, M. R., in preparation.

[2] Stormont, C., in preparation.

[3] Owen, R. D., *Science*, **102**, 400–401 (1945).

[4] Lillie, F. R., *Ibid.*, **43,** 611–613 (1916).
[5] Stormont, C., and Cumley, R. W., *J. Hered.*, **34,** 34–41 (1943).
[6] Ferguson, L. C., Stormont, C., and Irwin, M. R., *J. Immunol.*, **44,** 147–164 (1942).
[7] Holtman, D. F., *Ibid.*, **36,** 405–412 (1939).
[8] Holtman, D. F., *Ibid.*, **36,** 413–424 (1939).
[9] Landsteiner, K., *The Specificity of Serological Reactions*, Harvard University Press, Cambridge, Mass., 1945.
[10] Irwin, M. R., and Cumley, R. W., *Am. Nat.*, **77,** 211–233 (1943).
[11] Haldane, J. B. S., *Perspectives in Biochemistry*, edited by Needham and Green, Cambridge University Press, 1938, pp. 1–11.

*FURTHER STUDIES ON THE EFFECT OF INFRA-RED RADIATION ON X-RAY-INDUCED CHROMATID ABERRATIONS IN TRADESCANTIA**

By C. P. Swanson

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Communicated by K. Sax, March 21, 1949

The time limit within which x-ray-induced breaks in the chromosomes of dividing cells restitute, or recombine non-homologously, has not been established with any degree of exactitude, if, indeed, such a determination is possible. The problem has been variously attacked, but no single generalization adequately encompasses the divergent data obtained from such organisms as Tradescantia, Drosophila and maize. Even from the extensive x-ray studies which have been made on the microspore chromosomes of Tradescantia, the resultant conclusions drawn by the several investigators are only in partial agreement.

Time-intensity and fractional dosage studies on the microspore chromosomes of Tradescantia have led Sax to conclude that the broken ends remain in a condition permitting recombination or restitution for a period no longer than one hour.[1, 2] There is some indication that the time span may be as short as 20 minutes. The fact that temperature changes and centrifugation will modify the x-ray-induced frequency of aberrations in the same material *only* when used within certain time limits supports this conclusion.[3, 4] Data from experiments employing various radiations in combination provide additional supporting evidence. Ultra-violet, used as a pre-treatment, reduces the percentage of x-ray-induced aberrations in both Drosophila and Tradescantia.[5, 6] When used as a post-treatment on the pollen tube chromosomes of Tradescantia, the effectiveness of the ultra-violet in reducing the percentage of detectable aberrations diminishes as

the time interval between x-ray and ultra-violet radiation increases. The double (isochromatid) deletions appeared to be unaffected by post-treatment, and hence were considered to be formed immediately upon x-radiation; the chromatid exchanges showed no reduction in number when treated with ultra-violet one hour after the breaks were induced; the single chromatid deletions continued to be reduced in number for at least an hour after their initial induction.

In addition to the direct observational evidence discussed above, Lea and Catcheside[7] and Marinelli, *et al.*,[8] have contributed mathematical support. The former have concluded that primary chromatid breaks remain in a rejoinable condition for an average period of $3^1/_2$ minutes; the latter state that "90 per cent of the single chromosome breaks would heal within 115 minutes after their occurrence." Although the time discrepancies are large, neither hypothesis seriously refutes Sax's conclusion that the maximum duration of the rejoinable period for broken ends of chromosomes is between 20 to 60 minutes after the initial rupture has occurred.

On the other hand, the findings in Drosophila and maize give definitive proof that broken chromosomes need not undergo recombination or restitution within any limited interval of time other than that imposed by the reproductive mechanism of the organism. Both Muller[9] and Kaufmann[10] have demonstrated that x-ray-induced breaks in the mature sperm cells of Drosophila do not recombine until after the sperm has entered the egg in the process of fertilization, and in Muller's work, the time lapse for some breaks must have been of a month's duration. McClintock[11] has most ingeniously provided evidence that a similar condition holds for the mechanically induced broken ends of chromosomes in the sperms and eggs of maize, since recombination between paternal and maternal chromosomes can be observed to have taken place in the zygotic nucleus. Evidence of a somewhat different, though nonetheless relevant, character can be extracted from Sparrow's[12] study of the relative sensitivity of the various chromosome stages to x-rays. His data revealed that the chromosomes of Trillium, x-rayed at meiotic metaphase I, show few or no aberrations at metaphase I or anaphase I, the latter stage having a duration of at least one day. The aberrations are observed only at the first microspore metaphase. This can only mean that recombination has been very much delayed, taking place probably in interphase or during microspore prophase, no intervening meiosis II stages being present under the experimental conditions employed. One is consequently forced to the conclusion that the time interval between breakage and recombination will vary with the type of nucleus irradiated, and that the data from maize, Drosophila, Tradescantia and Trillium cannot be equated on any single temporal basis.

The use of infra-red radiation as a supplementary tool for modifying the rate of aberrations provides a means for reinvestigating the problem in

Tradescantia, a time interval of 21 hours being imposed between x-radiation and chromosome fixation. (This time limit is arbitrarily imposed to insure the detection of chromatid aberrations *only*, and to reduce possible variance in the rate of aberrations arising out of variations in the state of the nucleus at the time of radiation.) If it is assumed that *all* of the broken ends restitute or recombine within a certain period after x-radiation, then a

TABLE 1

FREQUENCY OF X-RAY-INDUCED CHROMATID ABERRATIONS WHEN INFRA-RED IS APPLIED AS A POST-TREATMENT. DATA IN %. 107 R

TIME IN HRS. AFTER X-RAY		TOTAL CHROMOSOMES	SINGLE DELETIONS	DOUBLE DELETIONS	EXCHANGES	TOTAL ABERRATIONS
1	x-ray control	1,914	4.02	8.41	1.05	13.48
	x-ray + infra-red	1,224	4.57	10.37	5.39	20.33
2	x-ray control	876	2.74	10.50	0.91	14.15
	x-ray + infra-red	1,122	3.12	10.78	5.70	19.60
4	x-ray control	798	2.88	8.27	1.75	12.90
	x-ray + infra-red	1,080	3.88	6.57	6.85	17.20
5	x-ray control	1,260	3.08	4.12	1.90	9.10
	x-ray + infra-red	1,506	4.51	8.20	6.50	19.21
9	x-ray control	1,548	3.23	5.55	3.00	11.78
	x-ray + infra-red	1,548	7.00	7.75	10.85	25.60
12	x-ray control	768	2.38	4.43	0.26	7.07
	x-ray + infra-red	1,080	3.14	6.66	11.11	20.91
15	x-ray control	1,452	2.50	4.44	0.55	7.49
	x-ray + infra-red	1,896	5.12	5.48	4.80	15.40
18	x-ray control	972	4.32	9.26	2.47	16.05
	x-ray + infra-red	864	5.10	6.48	5.50	17.08
Total	x-ray control	9,588	3.22 ± 0.18	6.55 ± 0.25	1.52 ± 0.12	11.29 ± 0.32
	x-ray + infra-red	10,320	4.69 ± 0.21	7.70 ± 0.26	7.05 ± 0.25	19.44 ± 0.39

stage should be reached after which the application of infra-red will no longer influence the rate of observable structural changes. Preliminary studies[13] have indicated that an infra-red treatment four hours after x-radiation still effects an increase in the rate of single deletions and exchange aberrations. The present report extends these data for the full period up to the time of chromosome fixation. In addition, these data as well as other findings to be presented enable us to examine further the hypotheses

which have been advanced to account for the mechanism of infra-red action as it affects the behavior of x-rayed chromosomes.

Materials and Methods.—A clonal line of *Tradescantia paludosa* Anders. and Woodson provided the experimental material. Fixation of the chromosomes was made at 21 hours after x-radiation, regardless of the time of application of the infra-red. All x-ray treatments consisted of unfiltered radiation from an industrial type of instrument, delivered at 80 kv. and 10 ma., and at a distance of 16 inches from the center of the tungsten target. Treatment time was $1^1/_2$ minutes, and the total dose, as measured by a Victoreen dosimeter, was 107 r. The buds to be treated with infra-red were placed in a large test tube, which was then rotated slowly for three hours in a beam of infra-red emitted by a commercial drying lamp operating at 110 volts and 250 watts. The radiation, before reaching the buds, was passed through a Corning glass filter No. 2540 opaque to visible wave-lengths of light but transmitting a high percentage of the infra-red from 0.75 to 4.0 microns. A stream of tap water constantly flowing over the rotating tube probably absorbed all infra-red wave-lengths longer than 1.5 microns, thus limiting the effective region to that portion of the spectrum between 0.75 and 1.5 microns. To insure adequate x-ray controls within each experiment, all the buds of any particular group were x-rayed simultaneously, whether they received infra-red or not. Only metaphase figures were scored for induced aberrations, and each figure in the "Total chromosomes" columns of tables 1 and 2 represents an analysis of 3 to 7 slides from acetocarmine smear preparations.

TABLE 2

FREQUENCY OF X-RAY-INDUCED CHROMATID ABERRATIONS WHEN A DELAY EXISTS BETWEEN THE TIME OF INFRA-RED AND X-RAY TREATMENTS. DATA IN %. 107 R

TREATMENT	TOTAL CHROMOSOMES	SINGLE DELETIONS	DOUBLE DELETIONS	EXCHANGES	TOTAL ABERRATIONS
X-ray	1752	2.74	5.13	0.91	8.78
Infra-red + x-ray:					
24-hour delay	1944	6.48	7.25	5.96	19.69
45-hour delay	1296	6.93	8.72	12.04	27.69
73-hour delay	1128	5.67	6.38	8.51	20.56
96-hour delay	1284	6.24	6.67	7.45	20.36

Experimental Results.—The data on chromatid aberrations derived from the infra-red post-treatment series of previously x-rayed buds are summarized in table 1. Each time-interval treatment is given with its own x-ray control data, and, for comparative purposes, each treatment of x-rays plus infra-red should be considered in terms of its own x-ray control. The fluctuations existing between the various control treatments make such comparisons necessary.

Considering each type of aberration separately, it is evident that the greatest increases are to be found in the exchange aberrations. Except for the 18-hour series which shows a difference significant only at the 5 per cent level, the remainder of the treatments show increases which are highly significant. The increases in single and double deletions resulting from infra-red post-treatment are not as striking. The former show increases which are consistently higher throughout all treatments, while the double deletions show decreases with infra-red in the 4- and 18-hour series. The increases nevertheless are highly significant for both types of aberrations whether the P values are calculated by standard error methods or by summation of the individual chi squares. There can be little doubt, therefore, that infra-red post-treatments can effectively modify the rate of detectable aberrations in Tradescantia throughout the entire prophase period. Considered in the light of Sax's extensive data from time-intensity, fractional dosage and temperature experiments on identical material which definitely point to a rather brief period during which recombination can occur, these increases were somewhat unexpected.

How permanent are the chromosomal effects of infra-red which lead to a greater observable rate of x-ray-induced aberrations? The quantum energies of infra-red within the spectral limits employed are not of sufficient magnitude to produce dissociation within most molecular systems, and the induced vibrational and rotational changes of electrons are of momentary duration. It has been suggested[14] that low-grade disturbances within the polymerized nucleoproteins may "sensitize" the chromosomes to subsequent treatments with x-rays, but our lack of knowledge of the chemical organization of the chromosome, and of the possible effect of the infra-red radiation upon such a chemical system makes any inquiry as to the mechanism involved a purely speculative one. Earlier studies on Tradescantia[15] indicated, however, that the induced changes, whatever their nature might be, are at least of a semipermanent character, for an interval of 21 hours between the infra-red and x-ray treatments led to no appreciable reduction in infra-red effectiveness. The data in table 2 indicate that the time interval between the two radiations may be extended to 96 hours without reducing the infra-red effect. The changes, once induced in the chromosomes, are apparently not reversible with time. The time interval employed, however, is not of sufficient length to encompass an entire mitotic cycle. Since the induced effect can be brought about in resting nuclei and persist until prophase, the stage at which chromatid breaks are presumed to be produced by x-rays, it is possible that the "sensitization" mechanism has as its basis some alteration induced in the processes which transform the chromosomes from a resting to a prophase state. Apart from subtle chemical changes which cannot be detected cytologically, the more conspicuous phenomena are the progressive dehydration, nucleination and coiling of the

chromosomes. That coiling is unaffected by the infra-red is reasonably certain; the length and diameter of the chromosomes appear the same whether infra-red has been applied or not. Little can be said, from a strictly cytological point of view, about dehydration of the chromosomes. On the other hand, if the infra-red interferes, in any way, with nucleic acid polymerization, and its attachment or detachment to the chromosome framework, a process which according to some cytologists stabilizes the chromosome during cell division, then one might theorize that any agent upsetting nucleination would tend to increase or decrease the fragility of the chromosome to x-radiation. As has been emphasized, however, whatever changes may occur in the chromosomes are not reflected in any morphologically detectable alteration of structure. The only consistent difference noted between control and infra-red-treated microspores is an increased percentage of cells observed dividing at an oblique angle. In the untreated microspores of Tradescantia, the spindle is oriented in the short axis of the cell. The presence of two large sap vacuoles, one at either end of the cell, makes this, however, the long axis of the cytoplasm. About 5 per cent of the cells normally lack one vacuole, thus producing a condition such that the long axis of the cytoplasm is oblique. The spindle orients itself accordingly. In all infra-red-treated buds, 25 per cent of the cells, on an average, possess an oblique division plane with an accompanying absence of one vacuole or both. The meaning of this observation in terms of an increased rate of chromatid aberrations is obscure, if indeed there is any correlation, unless one is dealing here with a viscocity change which affects both chromosomes and cytoplasm.

Discussion.—The interpretation placed upon the data presented in this report will depend upon the light in which they are viewed, i.e., whether the data are viewed in the light of the Kaufmann and Gay[16] hypothesis as to the mechanism of infra-red effect, or in the light of the hypothesis advanced by Swanson and Hollaender.[15] Consideration must also be given to the extensive data on x-rayed microspore chromosomes of Tradescantia, all of which point to the occurrence of recombination within a relatively short period of time after the initial induction of the primary breaks.

Kaufmann and Gay[16] consider that the production of primary breaks in the chromosomes of Drosophila sperm cells results *only* from the ionizing action of the x-rays. The additional infra-red radiation merely makes more of the breaks available for purposes of recombination, i.e., recombination is favored at the expense of restitution, but the infra-red does not of itself play any rôle in increasing the number of primary breaks. The principal basis upon which this hypothesis rests is the evidence that infra-red, while effectively modifying the rate of detectable recombinations, is without effect on the rates of induced dominant or recessive lethals, both presumably being single-hit phenomena. The second hypothesis,[15] which advanced the

opinion that the infra-red radiation was instrumental in increasing the actual number of primary breaks, was proposed in order to relate the data for both the pre- and post-treatments to a single mechanism. It was suggested that "if both infra-red and x-rays are capable of weakening the chromosome structure (this in addition to the normally realized x-ray breaks) in a manner which would not be detectable as distinct breaks, and which would not be realized unless further disturbed by the addition of the other type of radiation, then an additional increase in break frequency would be expected with both pre- and post-treatments with infra-red radiation." Suffice it to say, no critical evidence has yet been obtained which invalidates one or the other, or both, of the hypotheses. Physical data alone would suggest that infra-red energies are incapable of disrupting ordinary molecular bonds, but our lack of knowledge as to the effects of infra-red on macromolecular systems such as the chromosome nucleoproteins makes such considerations mere speculations.

Time-intensity,[1] fractional dosage,[2] temperature[3] and centrifugation[4] data fit into the interpretation that the time interval of recombination of x-rayed microspore chromosomes of Tradescantia is comparatively brief; it is to be measured in minutes and not in hours. The data in table 1, however, definitely indicate that infra-red post-treatments effectively increase the rate of all aberrations for a time interval between the x-ray and infra-red treatments even as great as 18 hours. To account for these data on the hypothesis of Kaufmann and Gay it would have to be assumed that, after normal recombination has taken place, there still remains a reservoir of primary breaks which do not restitute and which, under the influence of infra-red, are activated to a state wherein they will enter into recombination with other similarly activated breaks or be expressed as single deletions. It must also be assumed that this reservoir of breaks cannot be tapped by other environmental agents such as temperature and centrifugation. Such an explanation is not in good agreement with the time-intensity and fractional dosage experiments.[1, 2]

On the other hand, if it be assumed that the infra-red can raise the rate of primary breaks, the data presented here and earlier[15] become explicable and at the same time remain conformable to the data of Sax. An increase in both single deletions and exchange aberrations is to be expected on this basis, whereas a decrease in single deletions would be expected to accompany an increase of exchange aberrations, if only recombination were influenced by the infra-red, since the exchanges arise from single deletion types of breaks. It is, of course, entirely possible that both hypotheses are correct. Infra-red may induce primary breaks and also affect recombination. At the moment, however, we are inclined to feel that, in light of the Tradescantia data, the essential effect of the infra-red is to induce breaks. At the same time it is recognized that this explanation is less likely when

the Drosophila data are considered by themselves. Obviously more critical experiments are needed to test the two hypotheses.

It was earlier stated[15] that double deletions were unaffected by infra-red post-treatment, these being single-hit aberrations which were realized at the time of radiation with x-rays. In the light of the data presented, this view is no longer tenable.

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[1] Sax, K., PROC. NAT. ACAD. SCI., **25,** 225–233 (1939).
[2] Sax, K., *Cold Spring Harbor Symposia Quant. Biol.*, **9,** 93–101 (1941).
[3] Sax, K., and Enzmann, E. V., PROC. NAT. ACAD. SCI., **25,** 397–405 (1939).
[4] Sax, K., *Ibid.*, **29,** 18–21 (1943).
[5] Kaufmann, B. P., and Hollaender, A., *Genetics*, **31,** 368–376 (1946).
[6] Swanson, C. P., *Ibid.*, **29,** 61–68 (1944).
[7] Lea, D. E., and Catcheside, D. G., *J. Genetics*, **44,** 216–245 (1942).
[8] Marinelli, L. C., Nebel, B. R., Giles, N., and Charles, D. R., *Am. J. Bot.*, **29,** 866–874 (1942).
[9] Muller, H. J., *J. Genetics*, **40,** 1–66 (1940).
[10] Kaufmann, B. P., PROC. NAT. ACAD. SCI., **27,** 18–24 (1941).
[11] McClintock, B., *Ibid.*, **28,** 458–463 (1942).
[12] Sparrow, A. H., *Ibid.*, **30,** 147–155 (1944).
[13] Swanson, C. P., *Am. J. Bot.*, **34,** 12a (1947).
[14] Kaufmann, B. P., Hollaender, A., and Gay, H., *Genetics*, **31,** 349–367 (1946).
[15] Swanson, C. P., and Hollaender, A., PROC. NAT. ACAD. SCI., **32,** 295–302 (1946).
[16] Kaufmann, B. P., and Gay, H., *Ibid.*, **33,** 366–372 (1947).

THE LAST ENTRY IN GAUSS'S DIARY

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Herglotz, in his paper "Zur letzten Eintragung im Gaussschen Tageguch" [*Leipzig. Ber.*, *73*, 271–276 (1921)] proves by means of the complex multiplication of elliptic functions, the following conjecture of Gauss (entered in his Tagebuch on July 9, 1814):

For a complex primary prime modulus $\pi \equiv 1 \bmod (1-i)^3$ the number of solutions of the congruence (including the solutions $x, y \equiv \pm i, \infty\,;\ \infty, \pm i$):

$$x^2y^2 + x^2 + y^2 \equiv 1 \pmod{\pi}$$

is equal to the norm of $\pi - 1$. Dedekind verified this for $N(\pi) < 100$. We give a proof (Theorem 1 below) based on elementary number theory. NOTATION: p denotes a prime $\equiv 1 \pmod 4$; g is a primitive root of p;

(n/p) is the Legendre symbol of quadratic residuacity; θ is defined uniquely by $p = \theta^2 + \phi^2$, $\theta \equiv 1$ or 3 (mod 4) according as $p \equiv 5$ or 1 (mod 8); finally

$$S_m = \sum_{x=1}^{p-1} \left(\frac{x^4 - g^m}{p}\right) \qquad [m = 0, 1, 2, 3].$$

THEOREM 1:

$$S_0 = 2\theta - 2.$$

Proof: We observe that our theorem follows immediately from the 5 relations:

(A) $S_1 = -S_3$

(B) $S_0 + S_2 = -4$

(C) $S_0^2 + S_1^2 + S_2^2 + S_3^2 = 8p + 8$

(D) $S_0 \equiv 0, 4 \pmod 8$ $[p \equiv 5, 1 \pmod 8]$

(E) $p = \theta^2 + \phi^2$; $\theta \equiv 1, 3 \pmod 4$ $[p \equiv 5, 1 \pmod 8]$

(A) $$\sum_1^{p-1} \left(\frac{x^4 - g}{p}\right) = \sum_1^{p-1} \left(\frac{1 - gy^4}{p}\right) = -\sum_1^{p-1} \left(\frac{g^3 - g^4y^4}{p}\right) = -\sum_1^{p-1} \left(\frac{g^3 - t^4}{p}\right) \left[x = \frac{1}{y} = \frac{g}{t}\right]$$

(B) $$\frac{p-1}{4}(S_0 + S_1 + S_2 + S_3) = \sum_{a=1}^{p-1}\sum_{x=1}^{p-1} \left(\frac{x^4 - a}{p}\right) = \sum_{x=1}^{p-1}\sum_{a=1}^{p-1} \left(\frac{x^4 - a}{p}\right) = -(p-1)$$

(C) $$\frac{p-1}{4}(S_0^2 + S_1^2 + S_2^2) + S_3^2 = \sum_{a=1}^{p-1}\sum_{x=1}^{p-1}\sum_{y=1}^{p-1} \left(\frac{x^4 - a}{p}\right)\left(\frac{y^4 - a}{p}\right) = \sum_{x=1}^{p-1} \left\{\sum_{y=1}^{p-1}\sum_{a=1}^{p-1} \left(\frac{a^2 - a(x^4 + y^4) + x^4y^4}{p}\right)\right\}$$

Consider the sum inside the braces. If $y^4 \equiv x^4$, the innermost sum $= p - 2$; for the remaining $(p - 5)$ values of y the innermost sum is -2 since writing $f = \frac{1}{2}(x^4 - y^4)$ this sum easily transforms into

$$-1 + \sum_{a=1}^{p} \left(\frac{a^2 - f^2}{p}\right) = -1 + \sum_{t=1}^{p-1} \left(\frac{t}{p}\right)\left(\frac{t+2}{p}\right) = -1 + \sum_{t=1}^{p-1} \left(\frac{2t+1}{p}\right) = -2 \left[a = t + 1;\ t\bar{t} = 1;\ f \not\equiv 0\right].$$

(D) This follows at once by noting that x^4 has the same non-zero value mod p for exactly 4 incongruent values of x, and these x are equidistant from $\frac{1}{2}p$; further exactly 4 terms of S_0 vanish, while the rest are $+1$ or -1.

Similarly we prove:

THEOREM 2:

$$\sum_{x=1}^{p-1}\left(\frac{x^6-1}{p}\right) = -2 - 4\theta$$

where θ is defined uniquely by $p = \theta^2 + 3\phi^2$, $\theta \equiv 1 \pmod 3$ and p is a prime $\equiv 1 \pmod{12}$.

I should like to thank Professor Siegel (who found slightly more general results) and Professor Weyl for their interest in this note.

Note added in proof: Results similar to Theorems 1 and 2 have been found recently by E. Lehmer and A. L. Whiteman. Deuring, in a paper dedicated to Herglotz, finds more general results by advanced methods [see *Hamburg Abhandlungen*, **14**, 1941, 197–272].

ON THE THEOREM OF JENTZSCH

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1. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad \limsup_{n=\infty} |a_n|^{1/n} = 1, \tag{1}$$

be a power series with unit radius of convergence. Denote by

$$S_n(z) = \sum_{\nu=0}^{n} a_\nu z^\nu \qquad (n = 0, 1, 2, \ldots) \tag{2}$$

the partial sums of (1) and by Δ_n the distance of $z = 1$ from the set of roots of $S_n(z) = 0$. The classical result of R. Jentzsch[1] that every point of the circle of convergence is a limit point of zeros of partial sums may be written in the form:

$$\liminf_{n=\infty} \Delta_n = 0. \tag{3}$$

2. It is *impossible* to improve on (3) by exhibiting a sequence (φ_n) of positive numbers tending (however slowly) to infinity and such that

$$\liminf_{n=\infty} \varphi_n \Delta_n = 0$$

for *all* series (1). It is, however, possible to give explicit estimates of the rate with which (a suitable subsequence of) Δ_n approaches zero in terms of the moduli of the coefficients of (1) and of its behavior near $z = 1$. This

is an immediate corollary of more precise results, some of which are given below.

3. A function $f(z)$ regular in $|z| < 1$ (or in its intersection with a neighborhood of $z = 1$) is said to satisfy Condition C if

$$\log f(z) = o\left(\log \frac{1}{|1 - z|}\right)$$

as $z \to 1$ through some angle with vertice at $z = 1$ and one of whose sides contains the negative axis.[2]

Condition C is clearly satisfied if $f(z)$ is bounded and bounded away from zero as $z \to 1$ through the above angle, thus in particular if $f(z)$ is regular at $z = 1$ and $f(1) \neq 0$ or if $\sum a_n$ is convergent (or Cesaro summable of any order) to a sum different from zero.

4. First we give a result pertaining to the special case when the largest coefficients a_n are comparable to some power of n.

THEOREM 1. *Let $f(z)$ satisfy Condition C and, furthermore, let the coefficients of its Taylor series (1) satisfy*

$$-\infty < p = \limsup_{n=\infty} \frac{\log |a_n|}{\log n} < \infty. \tag{4}$$

Then there exist infinitely many n for which

$$S_n\left(1 + \frac{\zeta}{n} \log n\right) = 0 \tag{5}$$

has roots in the rectangle

$$-p - 1 - \delta < \xi = Re\{\zeta\} < -p + \delta, \qquad -\delta < \eta = Im\{\zeta\} < \delta, \tag{6}$$

δ being an arbitrary positive number.

Moreover, if $(n_\nu)_{\nu=1}^{\infty}$ is a sequence of increasing integers for which

$$\lim_{\nu=\infty} \frac{\log |a_{n_\nu}|}{\log n_\nu} = p, \tag{7}$$

then, for all ν sufficiently large, (5) for at least one of $n = n_\nu$ and $n = n_\nu - 1$ has roots in (6).

Thus we know of the existence of a subsequence of (Δ_n) for which the order of magnitude of Δ_n does not exceed $\frac{\log n}{n}$. Simple examples, e.g., $f(z) = (1 + z)^{-2}$, show this to be the correct order. Moreover, the location of some roots is known to within $\left(\frac{1}{2} + o(1)\right)\frac{\log n}{n}$ which, unlike the estimate of Δ_n, is independent of the value of p.

5. In the general case, when (6) is not assumed, there is the following analogous result.

THEOREM 2. *Let (1) satisfy Condition C then there exist infinitely many n (with $a_n \neq 0$) for which at least one of the two equations*

$$S_n\left(|a_n|^{-1/n} + \frac{\zeta}{n}\log n\right) = 0, \qquad S_{n-1}\left(|a_n|^{-1/n} + \frac{\zeta}{n}\log n\right) = 0 \quad (8)$$

has roots in the rectangle

$$-1 - \delta < \xi = Re\{\zeta\} < \delta, \qquad -\delta < \eta = Im\{\zeta\} < \delta, \quad (9)$$

δ being an arbitrary positive number.

Disregarding possible roots at $z = 0$, $|a_n|^{-1/n}$ is approximately the geometric mean of the moduli of the roots of $S_n(z) = 0$. It is remarkable that through consideration of distances from this geometric mean, instead of from $z = 1$, we have recovered in the general case the order $\frac{\log n}{n}$ encountered in *4*. This, however, does not yet yield any information, of the form considered in *2*, about the Δ_n. Indeed, as far as the statement of theorem 2 goes, it is conceivable that the values of n the existence of which is asserted are such that $\limsup |a_n|^{1/n} < 1$, and thus not even (3) is implied.

6. To overcome this failing it is necessary to specify further the values of n for which the assertion of theorem 2 is made. It turns out that these are those n for which the a_n are "large." Some sufficient conditions for this "largeness" are listed below:

An infinite sequence $(n_\nu)_{\nu=1}^{\infty}$ of increasing integers is an $\mathfrak{L}^1(a_n)$ sequence if it satisfies[3]

$$\max_{j \leq n_\nu}\left(\log |a_j| - \frac{j}{n_\nu}\log |a_{n_\nu}|\right) = O(\log n_\nu),$$

and

$$\sup_{j \geq n_\nu}\frac{1}{j}\log |ja_j| = \frac{1}{n_\nu}\log |n_\nu a_{n_\nu}| + o\left(\frac{\log n_\nu}{n_\nu}\right).$$

(n_ν) is an $\mathfrak{L}^2(a_n)$ sequence if to every $\epsilon > 0$ there corresponds a $\gamma > 0$ such that

$$\sup_{j \geq n_\nu}\left(\log |a_j| - \frac{j}{n_\nu}\log |a_{n_\nu}| + \gamma\frac{j}{n_\nu}\log n_\nu\right) \leq (\epsilon + o(1))\log n_\nu.$$

(n_ν) is an $\mathfrak{L}^3(a_n)$ sequence if there exist an integer N and a positive γ such that

$$\max_{N \leq j \leq n_\nu}\left(\frac{1}{j}\log |a_j| + \frac{1+\gamma}{j}\log j\right) = \frac{1}{n_\nu}\log |a_{n_\nu}| + \frac{1+\gamma+o(1)}{n_\nu}\log n_\nu.$$

Finally, an infinite sequence of increasing integers is an $\mathfrak{L}(a_n)$ sequence

if it is the union of three sequences each of which is either empty or an $\mathfrak{L}^k(a_n)$ sequence ($k = 1, 2, 3$).

It can be shown that whatever the series (1) there exist $\mathfrak{L}(a_n)$ sequences. It is also easily seen that if (n_ν) is an $\mathfrak{L}(a_n)$ sequence then

$$\lim_{\nu=\infty} |a_{n_\nu}|^{1/n_\nu} = 1. \tag{10}$$

Furthermore it is immediately verified that when (4) holds and (n_ν) satisfies (7) then it is an $\mathfrak{L}(a_n)$ sequence.

7. With this concept we have corresponding to (and including) the second part of theorem 1

THEOREM 3. *Let the conditions of theorem 2 be satisfied and (n_ν) be an $\mathfrak{L}(a_n)$ sequence, then for all $n = n_\nu$ with large ν at least one of the equations (8) has roots in (9).*

By choosing $\mathfrak{L}(a_n)$ sequences for which the approach of (10) to the limit is as fast as possible, explicit estimates of the type discussed in *2* of Δ_n are obtained.

8. Condition C is the only restriction on (1) in theorems 2 and 3. If it is not satisfied but there exists a real q for which $(z-1)^q f(z)$ satisfies Condition C the effect is a shifting of the rectangle (9) by $\frac{q}{n}\log n$. However, there are of course functions (1) which do not satisfy any restrictions of this type. Our methods yield, however, explicit estimates in the most general cases, we quote only the following result.

THEOREM 4. *Let (1) satisfy*

$$-\infty \leq \limsup_{n=\infty} \frac{\log |a_n|}{\log n} < -1, \tag{11}$$

let $\beta > 1$ be arbitrary and $(\beta_n)_1^\infty$ be such that $\beta_n \geq \frac{-1}{n}\log |a_n|$ and $\left(\beta_n - \frac{\beta}{n}\log n\right)_1^\infty$ is a downward convex sequence of positive numbers. Let $0 < \vartheta < 1$ be arbitrary and denote by M_n the maximum of $f(z)$ in $|z| \leq 1 - \vartheta\beta_n$, $|z-1| < \beta_n$, and let

$$\liminf_{n=\infty} \frac{1}{n\beta_n} \log M_n \leq 0.$$

Then

$$\liminf_{n=\infty} \frac{\Delta_n}{\beta_n} \leq 1. \tag{12}$$

Whenever (11) holds, sequences (β_n) satisfying all the above requirements can be found. Since $n\beta_n \to \infty$, (12) is an improvement of (3) of the type mentioned in *2*.

9. The previous results can be extended in yet another direction illustrated by

THEOREM 5. *Let (1) satisfy (4) with $p < -1$ and $(r_n)_1^\infty$ be any sequence of positive numbers subject to the only restriction $\log r_n = o(\log n)$ as $n \to \infty$. Then there exist infinitely many n for which $w = S_n\left(1 + \frac{\zeta}{n} \log n\right)$ maps (6) on a domain W_n over the w-plane completely covering the ring*

$$\frac{1}{r_n} < |w - \sum_{n=0}^{\infty} a_n| < r_n, \tag{13}$$

and, indeed, more than $\sigma \log n$ times, σ being a suitable positive number.

If instead of (6) larger domains are considered, (13) can also be replaced by larger domains. If, ultimately, (6) is replaced by $|z - 1| < \epsilon$ then (13) may be replaced[4] by $|w| < R_n$ where the R_n satisfy $\log R_n = o(n)$ as $n \to \infty$ and also $\sigma \log n$ by σn.

10. When the coefficients of (1) are such that (for $\mathfrak{L}(a_n)$ sequences) the approach to the limit in (10) is slow, e.g., $a_n = e^{\sqrt{n}}$, the rate of $\Delta_n \to 0$ is also slow, however the size of (6) can be diminished. As an example of yet another phenomenon we bring the following result on gap series.

THEOREM 6. *Let $(\lambda_n)_1^\infty$ be a sequence of increasing integers satisfying $\log (\lambda_n / (\lambda_n - \lambda_{n-1})) = o(\log n)$ as $n \to \infty$. Let $f(z) = \sum a_n z^{\lambda_n}$ satisfy*[5]

$$-\infty < \bar{p} = \limsup_{n=\infty} \frac{\log |a_n|}{\log n} < -1$$

and $\sum_{n=0}^{\infty} a_n \neq 0$. Then there are infinitely many n for which

$$S_{\lambda_n}\left(1 + \frac{\zeta}{\lambda_n} \log n\right) = 0$$

has roots in the circle

$$|\zeta - \frac{\bar{p}}{\lambda_n} \log n| < \frac{\delta}{\lambda_n} \log n, \tag{14}$$

δ being an arbitrary positive number.

The main improvement over theorem 1 is the appearance of $\log n$, instead of $\log \lambda_n$, in (14).

11. Condition C can be considerably weakened without impairing the order $\frac{\log n}{n}$, on the other hand if stronger assumptions are made this order can be improved; ultimately, when regularity or certain convergence type conditions are imposed, even to $\frac{1}{n}$. E.g.,

THEOREM 7. *Let (1) be regular and different from zero at $z = 1$, and its coefficients satisfy*

$$0 < A = \limsup_{n=\infty} |a_n| < \infty. \tag{15}$$

Then

$$\liminf_{n=\infty} n\Delta_n < B < \infty \tag{16}$$

Moreover, B may be chosen as a function of the single argument $|f(1)|d/A$ where d denotes the distance from $z = 1$ to the nearest singularity of $f(z)$.[6]

Similarly when a convergence type condition replaces regularity we have

THEOREM 8. *Let (1) satisfy (15) and $f(z)$ be bounded and bounded away from zero as $z \to 1$ through some angle with vertice at $z = 1$ and containing points in $|z| < 1$. Then if the sequence $S_n(1)$ is bounded, (16) again holds.*

Now, of course, instead of dependence on $f(1)$ and d which have no meaning in this case, B will depend on the bound of $S_n(1)$.

12. The shift mentioned at the beginning of *8* in the location of the roots of the partial sums when Condition C fails, is most clearly exhibited in the following theorem.

THEOREM 9. *Let (1) satisfy the conditions of theorem 7 except that $z = 1$ be a zero of order m of $f(z)$. Then*

$$\liminf_{n=\infty} n\Delta_n \left(1 - m\frac{\log n - \log\log n}{n}\right) < \infty, \tag{17}$$

where $\Delta_n(z_n)$ denotes the distance from z_n to the set of roots of $S_n(z) = 0$.

In particular we have in this case for the order of magnitude of (a suitable subsequence of) Δ_n the upper bound $\frac{\log n}{n}$. That all roots of the partial sums may indeed be as far from $z = 1$ as allowed by (17), is seen from examples such as $f(z) = \frac{1}{1+z} - \frac{1}{2}$.

13. The main tools used in establishing the above results are some of the simpler propositions of the theory of normal families or related inequalities of the theory of functions of a complex variable. Due to the general applicability of these tools, our results can be extended to much wider classes of developments than power series. In particular they can be carried over to general Dirichlet series (cf. reference 4) or even to Laplace-Stieltjes transforms. Also, instead of studying the partial sums of (1), it is possible to study in the same way the remainders or sums of blocks of terms from (1), etc., and similarly for more general developments.

Further results and proofs will be published elsewhere.

[1] *Acta Math.*, **41**, 219–251 (1918).

[2] The assumption that the angle contains points interior to the unit circle is sufficient; the stronger assumption in the text is made for normalization purposes, it affects the constants involved but not the order of magnitude.

[3] Unless otherwise stated, O and o refer to $\nu \to \infty$.

[4] Dvoretzky, A., *Studies on General Dirichlet Series* (Hebrew and English abstract), Jerusalem, 1941.

[5] This is, of course, quite different from (4).

[6] Only singularities of a principal branch need be considered.

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*ON DEVIATIONS BETWEEN THEORETICAL AND EMPIRICAL DISTRIBUTIONS**

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1. The determination of a theoretical distribution function $\sigma(u)$ by drawing a random sample is a familiar technique. The reliability of this procedure is based on considerations of the least upper bound of the absolute value of the difference between $\sigma(u)$ and the empirical distribution $\sigma_n(u)$, defined as $1/n$ times the number of sample elements falling below u. The theory requires delicate and powerful mathematical tools. Considerable simplification can, however, be achieved by taking samples whose size is not fixed but drawn from a Poisson population. This note is devoted mainly to the exposition of this idea. We also consider different measures of deviations and compare the theories for samples of fixed and random sizes.

2. Let $X_1, X_2, \ldots$ be independent random variables having the same continuous distribution functions $\sigma(u)$. Let furthermore $\psi_u(x)$ be 0 or 1 according as $x > u$ or $x < u$. The empirical distribution function $\sigma_n(u; X_1, \ldots, X_n) \equiv \sigma_n(u)$ is defined by the formula

$$\sigma_n(u) = \frac{1}{n} \sum_{j=1}^{n} \psi_u(X_j). \tag{1}$$

Kolmogoroff[1] has shown that

$$\lim_{n \to \infty} \text{Prob.} \left\{ \text{l.u.b}_{-\infty < u < \infty} |\sigma_n(u) - \sigma(u)| < \frac{\alpha}{\sqrt{n}} \right\} = P(\alpha), \tag{2}$$

where

$$P(\alpha) = \frac{\sqrt{2\pi}}{\alpha} \sum_{k=0}^{\infty} e^{-(2k+1)^2 \pi^2 / 8\alpha^2}. \tag{3}$$

An elementary proof of this and some related results was subsequently given by Feller.[2] Quite recently Doob[3] derived Kolmogoroff's result using the theory of stochastic processes. He pointed out that

$$P(\alpha) = \text{Prob.} \left\{ \underset{0 \leq u \leq 1}{\text{Max.}} |y(u)| < \alpha \right\}, \tag{4}$$

where $y(u)$ $(0 \leq u \leq 1)$ is a Gaussian process satisfying the conditions,

$$y(0) = y(1) = 0, \; E\{y(u)y(v)\} = \text{Min.}\,(u, v) - uv. \tag{5}$$

Since the process $y(u)$ is related in a simple way to the classical Wiener process the calculation of (4) can be reduced to a more or less standard calculation in the diffusion theory. Doob's justification of (4) is heuristic and he states that a rigorous proof would be quite delicate.

Independently of Doob, the present author also used the reduction to the process $y(u)$ in connection with a related problem (see § 4 of this note).

3. Let N be a random variable having Poisson's distribution with mean μ and independent of the collection $X_1, X_2, \ldots$. We define the modified empirical distribution function $\sigma^*_\mu(u)$ by the formula

$$\sigma^*_\mu(u) = \frac{1}{\mu} \sum_{j=1}^{N} \psi_u(X_j), \tag{6}$$

with the understanding that if $N = 0$ the sum is to be taken as 0. The result which corresponds to that of Kolmogoroff is

$$\lim_{\mu \to \infty} \text{Prob.} \left\{ \underset{-\infty < u < \infty}{\text{l.u.b.}} |\sigma^*_\mu(u) - \sigma(u)| < \frac{\alpha}{\sqrt{\mu}} \right\} = P^*(\alpha), \tag{7}$$

where

$$P^*(\alpha) = \text{Prob.} \left\{ \underset{0 \leq u \leq 1}{\text{Max.}} |x(u)| < \alpha \right\} = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} e^{-(2k+1)^2\pi^2/8\alpha^2} \tag{8}$$

and $x(u)$ denotes the Wiener process $(x(0) = 0)$. An intuitive justification of (8) is quite simple. We first note, following Kolmogoroff, that as long as $\sigma(u)$ is continuous, the probability on the left side of (7) is independent of $\sigma(u)$. Thus it is sufficient to restrict oneself to the case when $\sigma(u)$ is the uniform distribution, i.e., $\sigma(u) = u$, $0 \leq u \leq 1$.

Next, it is seen that the process

$$x_\mu(u) = \sqrt{\mu} \left\{ \frac{1}{\mu} \sum_{1}^{N} \psi_u(X_j) - u \right\}, \qquad 0 \leq u \leq 1 \tag{9}$$

is a process with *independent increments*[4] and

$$E\{x_\mu(u)\} = 0, \qquad E\{x^2_\mu(u)\} = u. \tag{10}$$

Moreover, as $\mu \to \infty$, the probability distribution of $x_\mu(u)$ becomes normal[5] with mean 0 and variance u. Thus it is natural to expect that the limiting properties of the process $x_\mu(u)$ will be those of a Gaussian process

$x(u)$ $(x(0) = 0)$ with *independent increments* and such that $E\{x(u)\} = 0$, $E\{x^2(u)\} = u$. These properties are characteristic of a Wiener process.

To justify (8) rigorously one needs to prove that

$$\lim_{\mu\to\infty}\lim_{\mu\to\infty} \text{Prob.}\left\{\max_{1\leq k\leq 2^r} |x_\mu\left(\frac{k}{2^r}\right)| < \alpha\right\} = \text{Prob.}\left\{\max_{0\leq u\leq 1} |x(u)| < \alpha\right\} \tag{11}$$

whereas the corresponding fact with the limiting processes interchanged (first $\mu \to \infty$ then $r \to \infty$) is trivial. Due to the fact that $x_\mu(u)$ is a process with *independent increments* (11) can be proved quite easily. In fact, one need only apply, with minor modifications, the reasoning given on page 295 of the note of Erdös and the present author, "On Certain Limit Theorems of the Theory of Probability."[7] The analogue of (11) for the process $y(u)$ is difficult and it is this step which is left out from Doob's considerations.

The deviation between the empirical and the theoretical distributions can be studied in various other ways. The following seems to be of some independent interest.

Let $V_a(x) = 1 - \psi_a(x)$ and consider the random variable

$$K_a = \int_{-\infty}^{\infty} V_a(\sqrt{\mu}(\sigma_\mu^*(u) - \sigma(u))\, l\sigma(u) = \int_0^1 V_a(x_\mu(u))du. \tag{12}$$

This random variable has a simple interpretation. The graph of $\sigma_\mu^*(u)$ is a step function and it is convenient to consider the jumps as vertical parts of the graph. K_a represents then the sum of lengths of those pieces of the vertical parts of the graph which lie above $\sigma(u) + a/\sqrt{\mu}$.

It is easy to prove that

$$\lim_{\mu\to\infty} \text{Prob.}\{K_a < \alpha\} = \text{Prob.}\left\{\int_0^1 V_a(x(u))du < \alpha\right\}. \tag{13}$$

A general theory of calculating probability distributions of Wiener functionals of the type $\int_0^1 V(x(u))du$ has been recently given by the present author.[6] For $a = 0$ one gets a particularly simple result (*loc. cit.*,[8] Example 2)

$$\lim_{\mu\to\infty} \text{Prob.}\{K_0 < \alpha\} = \frac{2}{\pi} \text{arc sin } \sqrt{\alpha}. \tag{14}$$

4. Results analogous to (14) can be also obtained for empirical distributions based on samples of fixed size. In this case

$$K_a = \int_{-\infty}^{\infty} V_a(\sqrt{n}(\sigma_n(u) - \sigma(u))d\sigma(u)$$

has the same interpretation as above, but

$$\lim_{n\to\infty} \text{Prob.}\left\{K_a < \alpha\right\} = \text{Prob.}\left\{\int_0^1 V_a(y(u))du < \alpha\right\}, \qquad (15)$$

where $y(u)$ is the Gaussian process defined by (5). Unlike in the case of the least upper bound the justification of (15) is not difficult. The calculations of the probability distributions of $\int_0^1 V_a(y(u))du$ can be reduced to considerations of the paper (*loc. cit.*[8]) but owing to the boundedness of V_a the theory can be simplified.

Let

$$Q_0(x, t) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$$

$$Q_{n+1}(x, t) = \int_0^t \int_{-\infty}^{\infty} \frac{e^{-(x-\xi)^2/2(t-\tau)}}{\sqrt{2\pi(t-\tau)}} V_a(\xi) Q_n(\xi, \tau) d\xi d\tau. \qquad (16)$$

It is easy to verify that

$$E\left\{\left(\int_0^1 V_a(y(u))du\right)^n\right\} = \sqrt{2\pi} n! Q_n(0, 1). \qquad (17)$$

Let furthermore

$$Q(x, t) = \sum_{n=0}^{\infty} (-1)^n v^n Q_n(x, t) \qquad (18)$$

and note that whenever $|V(\xi)| \leq M$ (in our case $M = 1$) one has

$$0 \leq Q_n(x, t) \leq \frac{(Mt)^n}{n!} Q_0(x, t). \qquad (19)$$

Thus $Q(x, t)$ is well defined and

$$0 \leq Q(x, t) \leq e^{Mt} Q_0(x, t).$$

Moreover,

$$\sqrt{2\pi}\, Q(0, 1) = E\left\{e^{-v\int_0^1 V_a(y(u))du}\right\}. \qquad (20)$$

From (16) and (18) it follows that

$$Q(x, t) + v\int_0^t \int_{-\infty}^{\infty} \frac{e^{-(x-\xi)^2/2(t-\tau)}}{\sqrt{2\pi(t-\tau)}} V_a(\xi) Q(\xi, \tau) d\xi d\tau = \frac{e^{-x^2/2t}}{\sqrt{2\pi t}} \qquad (21)$$

and from (19) that the Laplace transform

$$\varphi(x) = \int_0^{\infty} e^{-st} Q(x, t) dt, \qquad s > M,$$

exists.

Thus taking Laplace transforms of both sides of (21) we obtain

$$\varphi(x) + \frac{v}{\sqrt{2s}} \int_{-\infty}^{\infty} \varphi(\xi) V_a(\xi) e^{-\sqrt{2s}|x-\xi|} d\xi = \frac{e^{-\sqrt{2s}|x|}}{\sqrt{2s}} \qquad (22)$$

The *unique* solution of (22) can be shown to satisfy the differential equation[9]

$$\tfrac{1}{2}\varphi''(x) - (s + vV_a(x))\varphi(x) = 0, \qquad x \neq 0, \qquad x \neq a, \quad (23)$$

and the conditions

$$\lim_{x \to \pm\infty} \varphi(x) = 0, \qquad |\varphi'(x)| < A, \qquad \varphi'(-0) - \varphi(+0) = 2.$$

Moreover, $\varphi'(x)$ should be continuous for all $x \neq 0$.

In the simple $a = 0$ it follows that

$$\varphi(x) = \frac{\sqrt{2}}{\sqrt{s+v} + \sqrt{s}} e^{-\sqrt{2(s+v)}x}, \qquad x > 0$$

$$\varphi(x) = \frac{\sqrt{2}}{\sqrt{s+v} + \sqrt{s}} e^{\sqrt{2s}x}, \qquad x < 0,$$

and therefore

$$\varphi(0) = \int_0^\infty e^{-st} Q(0, t)dt = \frac{\sqrt{2}}{\sqrt{s+v} + \sqrt{s}} = \frac{\sqrt{2}}{v} \sum_{k=1}^{\infty} \binom{1/2}{k} \frac{v^k}{s^{k-1/2}},$$

thus

$$Q(0, t) = \frac{\sqrt{2}}{v} \sum_{k=1}^{\infty} \binom{1/2}{k} \frac{t^{k-1/2}}{\Gamma(k - 1/2)} v^k$$

and in particular,

$$Q(0, 1) = \frac{\sqrt{2}}{2\Gamma(1/2)} \frac{1 - e^{-v}}{v}$$

It finally follows from (20) that $\int_0^1 V_0(y(u))du$ is uniformly distributed in (0, 1).

* Work done under an ONR contract.

[1] "Sulla determinazione empirica di una legge di distribuzione," *Inst. Ital. Attuari, Giorn.*, **4**, 1–11 (1933).

[2] "On the Kolmogorov-Smirnov Limit Theorems for Empirical Distributions," *Ann. Math. Stat.*, **19**, 177–189 (1948).

[3] To appear in the *Ann. Math. Stat.*

[4] Hurwitz, H., Jr., and Kac, M., "Statistical Analysis of Certain Types of Random Functions," *Ann. Math. Stat.*, **15**, 173–181 (1944). It is shown in this paper that N having Poisson's distribution is not only sufficient but also necessary for $x_\mu(u)$ to have independent increments.

[5] *Loc. cit.*,[4] pp. 176–177. More general results in this direction were recently published by H. Robbins, "The Asymptotic Distribution of the Sum of Random Number of Random Variables," *Bull. Amer. Math. Soc.*, **54**, 1151–1161 (1948).

[6] *Loc. cit.*,[4] p. 177.

[7] *Bull. Am. Math. Soc.*, **52**, 292–302 (1946).

[8] "On the distributions of certain Wiener functionals," *Trans. Am. Math. Soc.*, **65**, 1–13 (1949).

[9] It is this equation which, in a different way and under more general conditions, is derived, *loc. cit.*[8]

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*ON INTERPOSITION AS A CUE FOR THE PERCEPTION OF DISTANCE**

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Interposition is usually defined[1–3] as the cue for the perception of the relative distances of two objects that results when one object partially obscures or overlaps the outline of another object. Such a definition is not satisfying, for it appeals to an unknown factor: the basis for discriminating between the object that obscures and the object that is obscured.

Brown[4] describes the cue in terms of completeness of outline: "We therefore perceive objects which are complete in outline as placed in front of similar objects which are incomplete." If it is added that knowledge of completeness of outline is a function of past experience, then Brown's specification of interposition becomes valid, albeit not fruitful. However, it probably does not apply in cases where there is no familiarity with the objects outlined; yet cues for relative position may remain in these cases.

Helmholtz,[5] in 1866, pointed out that even with objects whose forms are unfamiliar, "the mere fact that the contour line of the covering object does not change its direction where it joins the contour of the one behind it, will generally enable us to decide which is which." Helmholtz's specification is given in terms of essential stimulus characteristics, and it is the writer's belief that the statement specifies the cue in all cases, not merely when the objects are unfamiliar. The purpose of the present account is to state Helmholtz's description precisely and to point out some of its consequences.

Consider a projection of figures onto a plane perpendicular to the visual axis such that the nodal point of the eye is the point of projection. In such a perspective transformation all objects will be represented by simple closed curves. (Without any loss in generality only two figures will be treated here; the same reasoning may be extended to apply to any number of objects.)

A necessary condition for interposition to become effective as a cue is that the curves of the two objects in the plane of projection have a common boundary. A point at which the boundaries of the curves meet will be called a point of intersection. Clearly there will be two points of intersection for a common boundary.

Helmholtz's specification implies that the behavior of the functions represented by the closed curves at the points of intersection alone determines which object will be seen as nearer. More specifically Helmholtz's assertion may be taken to indicate that continuity of the first derivative of the object's contour at the points of intersection is the sole determiner of relative distance. (A first derivative $f'(x)$ of a function $f(x)$ is continuous at a point p if and only if for any $\epsilon > 0$, there exists a δ such that $|f'(x) - f'(p)| < \epsilon$ in the interval $|x - p| < \delta$.)[6]

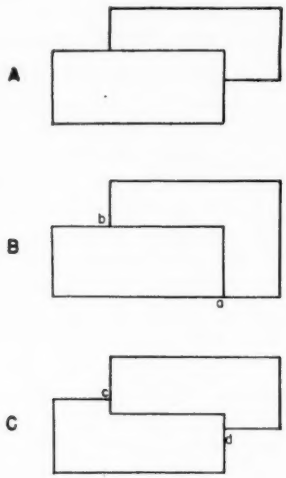

FIGURE 1

Examples of the operation of interposition as a cue for the perception of relative distance. *A* and *B* give obvious cues. The cues in *C* result in an ambiguity.

This formulation predicts that what happens at one point of intersection is independent of what happens at the other. That is to say, both points of intersection may give the same cue, both may offer no cue, only one may give a cue, or each may provide a cue which contradicts the other. If, at a point of intersection, the contours of both objects are the same with respect to continuity of their first derivatives, no cue will be provided at that point of intersection; if this occurs at both points of intersection, interposition will afford no cue for the relative distances of the two objects.

The cases in which interposition provides a cue may be classified in the following manner:

Case I. One boundary's first derivative is continuous at both points of intersection; the other boundary has a discontinuous first derivative at both points of intersection. This case is illustrated in figure 1 (*A*), in which the

figure whose boundary has a continuous first derivative at the points of intersection, i.e., the left-hand figure, is seen as nearer; the figure whose boundary has a discontinuous first derivative, i.e., the right-hand figure, is seen as farther away. In figure 1 (*A*), the same cue is provided at both points of intersection.

Case II. One function possesses a continuous first derivative at one point of intersection and a discontinuous one at the other, while the other function's first derivative is discontinuous at both points of intersection. This case is shown in figure 1 (*B*). Here the point of intersection at which both first derivatives are discontinuous, point *a*, gives no cue. However, at point *b*, one function (that represented by the outline of the right-hand figure) has a discontinuous first derivative, and the other function (that represented by the outline of the left-hand figure) has a continuous first derivative. Thus, the left-hand figure is seen in front of the right-hand figure.

Cases I and II give familiar illustrations of interposition.

Case III. The first derivative of each function is continuous at one point of intersection and discontinuous at the other, and each derivative is continuous at that point of intersection at which the other is discontinuous. This case is illustrated in figure 1 (*C*). Here a cue is provided at each point of intersection, but the two cues contradict each other. The point of intersection at *c* indicates that the right-hand figure is nearer, while that at *d* indicates that the left-hand figure is nearer. Thus, the resulting cues are in conflict.

Summary.—Helmholtz's specification of interposition is reformulated. It is asserted that: (1) interposition can provide a cue only at the points where the outlines of two objects meet; and (2) the object whose contour possesses a continuous first derivative at those points will be seen as nearer than an adjacent object whose first derivative is discontinuous at those points. Some consequences of this formulation are discussed.

* This account was prepared under Project NR 142–404, Contract N6onr–271, Task Order IX, between Columbia University and the Office of Naval Research, U. S. Navy.

[1] Boring, E. G., *Sensation and Perception in the History of Experimental Psychology*, Appleton-Century, New York, 1942, p. 264.

[2] Dimmick, F. L., "Visual Space Perception," in Boring, E. G., Langfeld, H. S., and Weld, H. P. (Eds.), *Foundations of Psychology*, John Wiley, New York, 1948, p. 298.

[3] Munn, N. L., *Psychology*, Houghton Mifflin, New York, 1946, pp. 352–354.

[4] Brown, W., "The Perception of Spatial Relations," in Boring, E. G., Langfeld, H. S., and Weld, H. P. (Eds.), *Psychology*, John Wiley, New York, 1935, p. 216.

[5] Helmholtz, H. von, *Physiological Optics*, Vol. III, Optical Society of America, 1925, pp. 283–284.

[6] To be continuous at a point, the first derivative must be defined at that point. Accordingly, in the following examples given in figure 1, the coördinate axes must be taken in such a way that neither is parallel to the sides of the figures containing a point of intersection.

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